THE
MATHEMATICAL GAZETTE.

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LONDON:
GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN,
AND BOMBAY.

VOL. IV. OCTOBER, 1907. No. 66.

ELEMENTARY ANALYSIS AT THE LEICESTER MEETING OF THE BRITISH ASSOCIATION.

AT the meeting of Section A on Monday, Aug. 5, Professor A. Lodge took the chair while the President, Professor Love, read a paper on the Exponential Theorem. He remarked that a simple, rigorous, and systematic treatment of the exponential theorem was required. The question presented itself naturally when the student, who had commenced the differential calculus, tried to differentiate $\log_{10} x$.

Inasmuch as

$$\frac{\log_{10}(x+h)-\log_{10}x}{h}=\frac{1}{x}\log_{10}\left(1+\frac{h}{x}\right)^{\frac{x}{h}},$$

the limit of $\left(1+\frac{1}{n}\right)^n$, when n increases without limit, forces itself into notice.

Numerical work with the log book suggests that the limit exists and is approximately 2·7. Assuming the binomial theorem for a positive integral index, it may be shewn that $\left(1+\frac{1}{n}\right)^n$ increases as the positive integer n increases, but is always less than 3.

In extending the conclusion to the case when n varies *continuously*, assuming irrational as well as rational values, a meaning must be assigned to powers with irrational exponents; which may be done at this stage, and as follows.

If A is any real positive number, a and b rational numbers, and x any number rational or irrational lying between a and b, then A^x lies between A^a and A^b.

Concluding that $\lim\limits_{n=\infty}\left(1+\frac{1}{n}\right)^n$ is a definite number between 2 and 3 we call this number e. By the process of differentiation indicated above we find $\frac{de^x}{dx}=e^x$, so that e^x has a differential coefficient which is continuous.

Consider now the function

$$\phi(x)=e-e^x-(1-x)e^x-\frac{(1-x)^2}{\underline{|2}}e^x\ldots-\frac{(1-x)^{n-1}}{\underline{|n-1}}e^x-\frac{(1-x)^n}{\underline{|n}}R. \quad \ldots\ldots(1)$$

This expression vanishes when $x=1$, and also when $x=0$, if

$$\frac{R}{\lfloor n}=e-1-1-\frac{1}{\lfloor 2}\dots-\frac{1}{\lfloor n-1}. \quad \dots\dots(2)$$

Hence $\phi'(x)$ must vanish for some value of x between 0 and 1, say when $x=a$.

But by differentiation

$$\phi'(x)=\frac{(1-x)^{n-1}}{\lfloor n-1}(-e^x+R).$$

Therefore $R=e^a$, and by (2)

$$e=1+1+\frac{1}{\lfloor 2}\dots+\frac{1}{\lfloor n-1}+\frac{e^a}{\lfloor n}, \quad 0<a<1.$$

The last term is less than $\frac{3}{\lfloor n}$, and so e may be computed to any required extent by the formula $1+1+\frac{1}{\lfloor 2}\dots+\frac{1}{\lfloor r}$.

Similarly, if we put

$$\psi(x)=e^b-e^x-(b-x)e^x-\frac{(b-x)^2}{\lfloor 2}e^x\dots-\frac{(b-x)^n}{\lfloor n}R,$$

and

$$R=\frac{\lfloor n}{b^n}\left[e^b-1-b-\frac{b^2}{\lfloor 2}\dots-\frac{b^{n-1}}{\lfloor n-1}\right]$$

we see that $\psi(x)$ vanishes when $x=b$ and $x=0$.

Hence

$$\psi'(x), \text{ or } \frac{(b-x)^{n-1}}{\lfloor n-1}(R-e^x)$$

vanishes when $x=a$ say, where $0<a<b$, so that we find $R=e^a$ and

$$e^b=1+b+\frac{b^2}{\lfloor 2}+\dots\frac{b^{n-1}}{\lfloor n-1}+\frac{b^n}{\lfloor n}\cdot e^a.$$

The result is the exponential theorem.

In this method e is introduced as the value of a limit which stands in our path and must be dealt with; the exponential theorem is proved by the same process which is used to prove Taylor's theorem, and the theory of infinite series is entirely avoided.

Dr. Young admired the ingenuity of Professor Love's method, but thought that it was artificial and savoured of the "calculus dodging" devices formerly in vogue at Cambridge. He thought that infinite series ought to be dealt with at an early stage and that the difficulties to be faced were not greater than those of irrational numbers and of a limit which Professor Love assumed to have been overcome.

Mr. C. S. Jackson admitted that the latter part of Professor Love's method looked artificial; but after all it was the method the student would have to apply subsequently to Taylor's theorem: and Professor Love certainly *started* the subject in the most natural way. The less beginners were troubled with questions of convergency of infinite series the better. This was proved by the historic fact of the very late evolution of the theory of convergency, which might be illustrated by a quotation from the *Encyclopédie des Sciences Mathématiques.*

"The mathematicians of the XVIII. century, when they gave themselves any concern at all about convergence, were satisfied, notwithstanding the example of the harmonic series given by John and James Bernoulli, as soon as they saw that the n^{th} term tended to zero, as n increased indefinitely. Lagrange himself does not escape this error, which indeed he explicitly adopted."

Dr. Hobson agreed that Professor Love's method, though at first sight artificial, was one which the student would have to understand. We still attached too much importance to the power of rapidly reproducing book-work in the examination room. It was quite possible for a student to have got all the good out of a topic, and to have thoroughly grasped a course of reasoning, and yet be unable to write it out against time. He would like the exponential theorem, and other theorems too, to be presented in several ways and from different points of view, but do not let a student be made to remember them all.

Professor A. Lodge thought that Professor Love's method was difficult, however suitable for the better students.

On Tuesday, Aug. 6th, Professor Love, President of the Section, in the chair, Dr. Young read a paper on the introduction of the idea of Infinity. [This is the paper we hope he will offer to the *Gazette*.]

Professor Forsyth observed that in his undergraduate days the dictum of Professor Lamb, that even in mathematics something must be risked, was adopted to its fullest extent. Later, when he was dealing with the functions of a complex variable he still risked a good deal. Dr. Young was one of a band of workers who were dealing faithfully with the theory of functions of a real variable, but he expected still to have to risk something.

The word infinity is used in a great variety of senses—the point at infinity, the line at infinity, countable infinities, infinity in the sense that $\frac{1}{0}$ was infinity. So was $\frac{2}{0}$. Were they the same infinity?

He would offer a suggestion to teachers of mathematics. Let them be cautious how they talked about infinity. As a boy he was taught geology, and found that glacial action was made to account for everything, until the suspicion crossed his mind that glacial action, whatever its geological character, was very convenient for the teacher. Do not let a teacher take refuge in infinity so regularly as to lead a boy to form the theory that infinity was a way round difficulties for the use of teachers who could not explain.

Mr. C. O. Tuckey's paper was read by Professor Lodge.

Throughout the meeting time was the enemy. This has led to growling in various quarters and to suggestions for the further subdivision of Section A—a course regrettable if necessary, because it will increase the desirability of being a bird, so as to be in two places at once. A universal recollection of Wellington's advice to speakers, "Wait until you have something to say, get up and say it and then sit down," would have materially shortened some of the earlier sessions. Whether a rigorous time limit of twenty minutes to an opener and five minutes to each subsequent speaker would not be preferable to further subdivision of Section A is a question. At any rate, that papers written by request should be cut out or cut down for lack of time is on all grounds to be regretted.

PERSPECTIVE THROUGH THE STEREOSCOPE.*

In the twelve slides described in this paper the *Picture Plane* is that determined by the two thickened vertical lines which appear in each diagram. It will be called the P. Pl.

*These slides were exhibited at the General Meeting held in January, 1906, by means of stereoscopes lent for the occasion by Messrs. Underwood and Underwood, 104 High Holborn. They have announced the sale price at 8s. 6d. the set of 12. The reproductions here are not in Central-, but in Parallel-Perspective.

Similarly the *Ground Plane* is that determined by the two thickened horizontal lines which appear in each diagram. It will be called the G. Pl.

The *Intersection* is the line in which the G. Pl. cuts the P. Pl.

E is the eye.

I is the foot of the perpendicular EI from E to the P. Pl. Hence I is the point variously called by writers on perspective as the Eye point, the *centre of the picture, the point of sight*, or *centre of vision.* Here it will be called the I-point.

EI is the line called the *Line of Direction* or *Principal Visual Ray.* Its length is the *Distance of the Picture.* The line through I in which the plane through E parallel to the G. Pl. cuts the P. Pl. is here called the I-line and is obviously parallel to the Intersection.

J is the foot of the perpendicular EJ from E to the G. Pl. It is here called the J-point.

The line through J in which the plane through E parallel to the P. Pl. cuts the G. Pl. is here called the J-line, and is also obviously parallel to the *Intersection.*

Note that EJ is parallel to the Picture Plane.

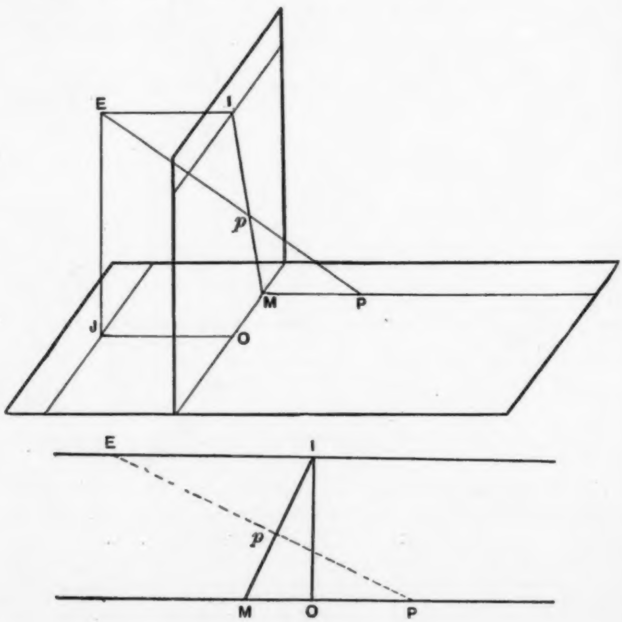


FIG. 1.

O is the point in which the plane IEJ cuts the Intersection. Hence IO in the P. Pl. is equal and parallel to EJ, and OJ in the G. Pl. is equal and parallel to IE.

Slide I. Here PM is any straight line drawn in the G. Pl. perpendicular to the Intersection. (Upper diagram in Fig. 1.)

Hence MP is perpendicular to the Picture Plane and therefore parallel to EI.

Hence EI, MP, lie in a plane which cuts the P. Pl. in the straight line IM.

But EP lies in the plane through EI, MP. Therefore p lies in the line of section IM. Hence

Theorem (1) *Any line on the ground plane perpendicular to the Intersection is represented on the picture plane by a straight line passing through the I-point.*

Again, by similar triangles,

$$\frac{Ip}{pM}=\frac{EI}{MP}.$$

Hence (First construction)

To find the point p on the P. Pl. corresponding to a given point P on the G. Pl. Take IE along the I-line equal to the *Distance of the Picture* and MP in the opposite sense along the Intersection equal to the known distance of P from the P. Pl. Join EP cutting IM in p.

The argument and construction applies to any other horizontal plane.

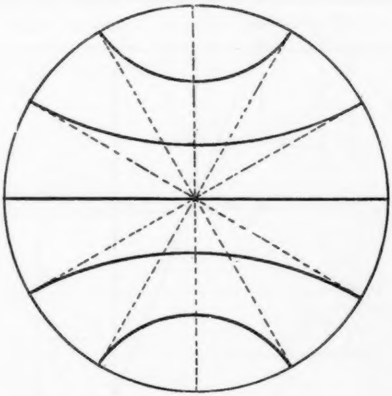


FIG. 2.

In Slide II. (not reproduced here) Theorem (1) is illustrated by a set (or *pencil*) of perpendiculars like MP and the set (or *pencil*) corresponding to them on the Picture Plane all passing through I.

If P moves along any one of the lines like MP continually further and further from the Picture Plane, its corresponding point p continually approaches but never actually reaches the eye point I. Hence I is called the vanishing point of the set of lines like MP.

An interesting case of the above theorem is afforded by either the *Stereographic* or the *Globular Projection* of the surface of a sphere on the plane of a meridian.

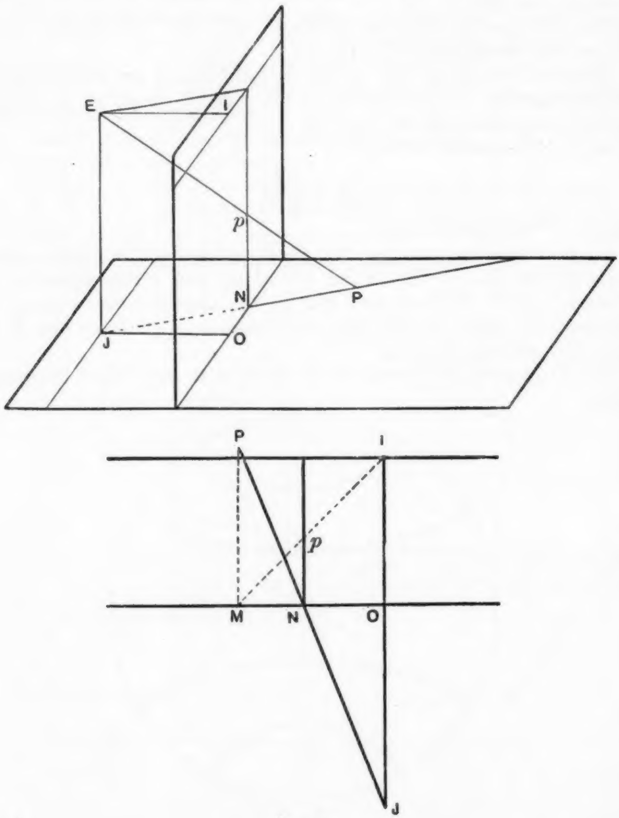


Fig. 3.

In each of these the tangents to the (projected) Parallels of Latitude at their extremities all pass through the centre of the meridian circle whose plane is chosen as the P. Pl.

For the Point of Sight *I* is at the centre of that meridian, and the original tangents are all perpendicular to the meridian.

Slide III. Here *NP* is any straight line in the G. Pl. passing when produced through *J*. (Upper diagram of Fig. 3.)

Hence the plane *EJP* cuts the P. Pl. in a straight line *Np* parallel to *EJ* and therefore perpendicular to the Intersection. Hence

Theorem (2) *A straight line NP on the G. Pl. which passes when produced through the J point is represented on the P. Pl. by a straight line Np perpendicular to the Intersection.* Hence (Second construction)

To find the point *p* on the P. Pl. corresponding to a given point *P* on the G. Pl.

Take *OJ* along *OI* equal to the *Distance of the Picture*, and in the opposite sense *MP* perpendicular to the *Intersection* equal to the known distance of *P* from the P. Pl. Join *JP* cutting the *Intersection* in *N*. (Lower diagram.)

Then the perpendicular to the *Intersection* through *N* cuts *IM* in *p*.

Note (1) If *IE* be drawn equal to the Distance of the Picture in the same sense as *OJ*, then *E, P, p* are in a straight line.

(2) The positions of *J, P, E* in this construction are those which the points so lettered on the slide would take if *OJ, MP, IE* were rotated in the same sense through a right angle about *O, M, I* into the P. Pl.

(3) This rotation might take place in either sense, *i.e. OJ, MP, IE* might be drawn in each in the opposite sense to that chosen.

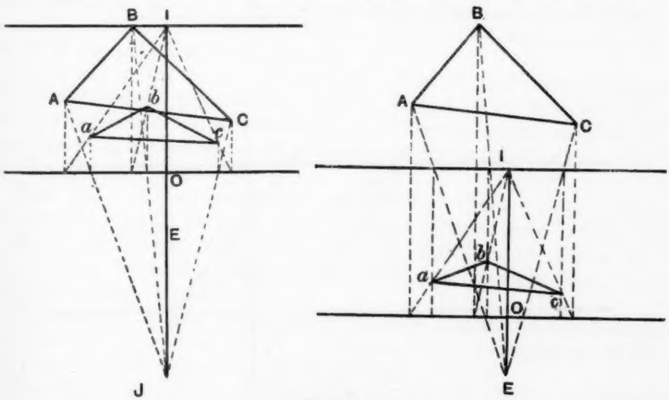


FIG. 4.

In Slide IV. (not reproduced here) Theorem (2) is illustrated by a set (or *pencil*) of lines on the G. Pl. passing when produced

through J and the set (or *pencil*) corresponding to these on the P. Pl. all perpendicular to the Intersection.

The subjoined figures shew the application of the Second construction to find the triangle abc in the P. Pl. corresponding to a given triangle ABC in the G. Pl. and a modification of it which is sometimes useful in practice (Fig 4).

In the first of the two the construction is merely that of the last section, *i.e.* the triangle ABC is drawn in the P. Pl. in the position it would occupy if rotated with the G. Pl. about the Intersection. In the second it is drawn in the position it would occupy if it first underwent this rotation and

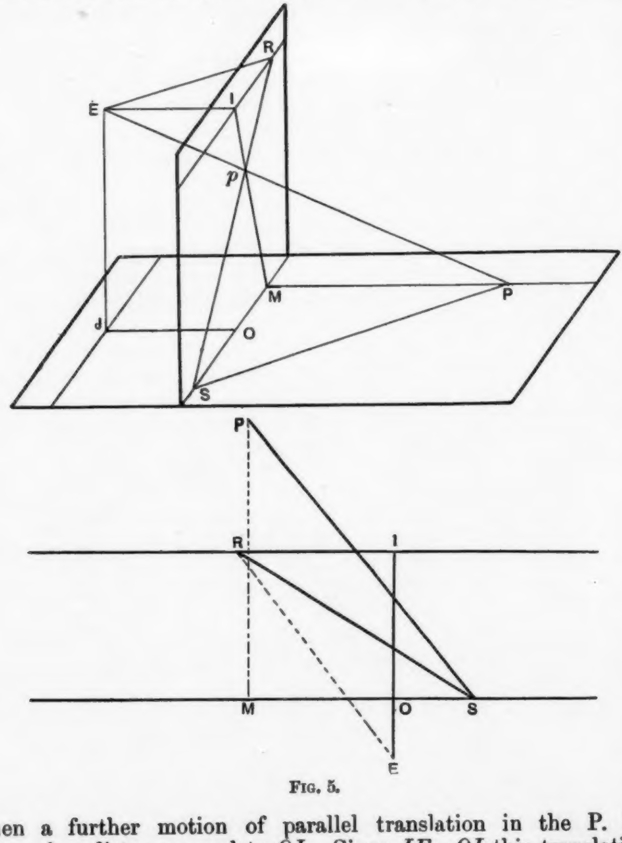


Fig. 5.

then a further motion of parallel translation in the P. Pl. through a distance equal to OI. Since $JE=OI$ this translation

makes *J* coincide with *E* and the Intersection with the vanishing line, and we use the points in which *EA, EB, EC* cut the Vanishing Line instead of those in which *JA, JB, JC* cut the Intersection.

Slide V. Here *SP* is any line in the Ground Plane. *ER* is the parallel to it through *E* and therefore lying in the plane *EIR* which cuts the Picture Plane in the *I-Line* or Horizontal Line. (Upper diagram in Fig. 5.)

Since *ER* is parallel to *SP*, they lie in a plane which cuts the Picture Plane in the line *RS*. Hence

Theorem (3) *Any line on the Ground Plane in a direction parallel to ER is represented on the Picture Plane by a straight line passing through R.* Hence (Third construction)

To find *p* on the P. Pl. corresponding to a point *P* on the G. Pl. given by its perpendicular *MP* to the Intersection and some other line *PS* through it. (Lower diagram.)

Draw *MPS, EIR* in the P. Pl. in the positions they would occupy if rotated into it about the Intersection and the *I*-line. Join *RS* cutting *IM* in *p*.

Note that this construction is the same as the First if *PS* is inclined at 45° to the Intersection.

In Slide VI. (not reproduced here) Theorem (3) is illustrated by a set (or *pencil*) of lines all parallel to *ER* and the set (or *pencil*) corresponding to them on the Picture Plane all passing through *R*.

If the point *P* move along any one of the pencil of lines like *SP*, farther and farther away from the Picture Plane, its corresponding point *p* will continually approach but never actually reach the point *R*. Hence *R* is called the *vanishing point* of the pencil of lines like *SP*.

Since any point on the *I*-line is the Vanishing Point of some pencil of lines, such as *SP* in the G. Pl., the *I*-line is called the *Vanishing line of the Ground Plane.* For the same reason it is also the vanishing line for all planes parallel to the G. Pl., *i.e.* for all Horizontal Planes.

Slide VII. Here the line on the Ground Plane passes when produced through a point *L* on the *J*-line. (Upper diag. in Fig. 6.)

Since the plane *EJL* is parallel to the Picture Plane its intersection with the plane *ELQ* will be a straight line *RQ* parallel to *EL*. Hence

Theorem (4) *A line which passes through a point in the J-line corresponds to a line in the Picture Plane parallel to that joining the eye to the point in the J-line.*

Hence the following construction in the P. Pl. for drawing the line corresponding to a given straight line in the G. Pl.

Draw *E, J, L* in the P. Pl. in the positions they would have if *IE* were rotated about the *I*-line and the G. Pl. were rotated,

in the same sense, about the Intersection, into the P. Pl. Join EL and draw QR parallel to LE. (Lower diagram.)

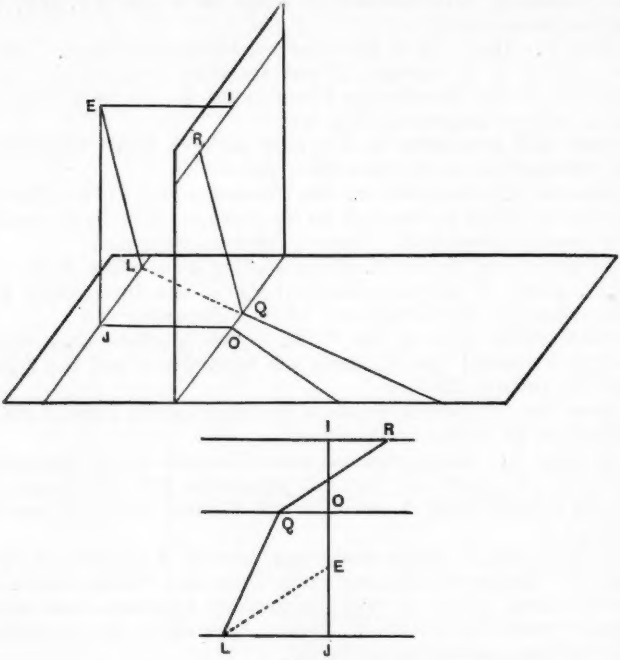


FIG. 6.

In Slide VIII. (not reproduced here) Theorem (4) is illustrated by a set (or *pencil*) of lines in the Ground Plane all passing through the same point L in the Foot line and the set (or *pencil*) corresponding to them on the Picture Plane all parallel to EL. *(To be continued.)*

MATHEMATICAL NOTES.

243. [K. 2. c.] *Notes on the Nine-Point Circle.*

1. D, E, F being the mid points of BC, CA, AB, to determine a point P on the nine-point circle such that

$$PE = PD + PF \quad (a > b > c).$$

Bisect the arcs DD', EE', FF' in α, β, γ.

Let X, Y, Z be the points of contact of the in-circle, and let the sides of XYZ cut the corresponding sides of DEF in $U_1 U_2 U_3$, so that

$$De' = DX = \tfrac{1}{2}(b-c),$$
$$Fe = FZ = \tfrac{1}{2}(a-b),$$

and

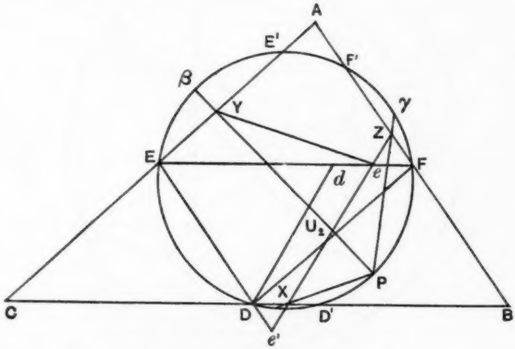
$$Ee = Ee' = \tfrac{1}{2}b.$$

Join βU_2, cutting the nine-point circle in P.

Then $$\angle DPU_2 = \angle FPU_2, \text{ standing on equal arcs ;}$$
$$\therefore\ DP : FP = DU_2 : FU_2 = DX : Fe$$
$$= b - c : a - b.$$

Now $$PE \cdot \tfrac{1}{2}b = PD \cdot \tfrac{1}{2}a + PF \cdot \tfrac{1}{2}c, \quad \text{(Euc. VI. D)}$$
and $$(a-c)b = (b-c)a + (a-b)c\ ;$$
$$\therefore\ PD : PE : PF = b - c : a - c : a - b\ ;$$
$$\therefore\ PE = PD + PF.$$

The point P is evidently unique, so that aU_1 and γU_3 also pass through it, bisecting the angles EPF, EPD externally.



2. The nine-point circle and the in-circle touch at P.

Draw Dd parallel to XZ, so that
$$Fd = \tfrac{1}{2}(a-c),$$
and $$dDF = \tfrac{1}{2}(A+C) - C = \tfrac{1}{2}(A-C).$$

Let $\beta U_2 P$ cut EE' in Y'.

Then, since $$U_2PF = \tfrac{1}{2}(A+C) = Eee'\ ;$$
$$\therefore\ eU_2PF \text{ is cyclic ;}$$
$$\therefore\ ePF = eU_2F = \tfrac{1}{2}(A-C)\ ;$$
$$\therefore\ Y'Pe = Y'PF - ePF = \tfrac{1}{2}(A+C) - \tfrac{1}{2}(A-C)$$
$$= C = Y'Ee\ ;$$
$$\therefore\ EY'ePe' \text{ is cyclic ;}$$
$$\therefore\ EeY' = EPY' = \tfrac{1}{2}(A-C) = dDF.$$

Also $$Y'Ee = dFD,$$
and $$Ee = \tfrac{1}{2}b = DF.$$
$$\therefore\ EY' = Fd = \tfrac{1}{2}(a-c)\ ;$$
$$\therefore\ Y' \text{ coincides with } Y.$$

But the tangents at Y, β are parallel ;

$\therefore$ βYU_2P passes through a centre of similitude.

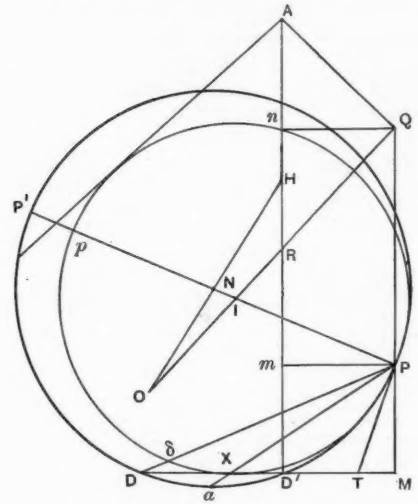
So for aX, γZ ; $\quad\therefore$ P is a centre of similitude, so that the circles touch at P.

3. Let $OIX=\phi$, $OI=f$, $NI=d$,

so that $f^2=2Rd$.

And $\sin\phi=DX/OI=R(\sin B-\sin C)/f$.

$\cos\phi=(r-R\cos A)/f=R(\cos B+\cos C-1)/f$.



Let DP cut the in-circle at δ.

Then $DP : D\delta = P'P : P'p = R : 2d$;

$\therefore\ DP^2 : DP \,.\, D\delta = R^2 : 2Rd$;

$\therefore\ DP/R = DX/f = \sin\phi$;

$\therefore\ DP'P=\phi=PD'T$;

$\therefore\ P'D=IP'\cos\phi=R^2(\cos B+\cos C-1)/f$.

4. Draw AQ $\perp$r. to OI.

Since $PD'M=\phi=OIX=ORD'$,

and $DPD'=B-C=OAR$;

$\therefore\ PDD'=AOR=\psi$.

Hence $D'm=PM=PD\sin PDD'$

$=R\sin\phi\,.\,\sin\psi$

$=AQ\sin\phi=An$,

and $D'M=PM\cot\phi=Qn$;

$\therefore$ PQ is parallel to AD', and $AQ=D'P$.

5. Draw PlL perpendicular to EF.

Then DL is parallel to the Simson lines of P with regard to DEF.

Now $LDD' = D'PM = 90 - \phi$, so that DL, and therefore the Simson line, is parallel to OI.

Let YZ cut DF, DE in d, d', so that

$$Dd = Dd' = \tfrac{1}{2}a;$$

and let XY cut FD, FE in f, f',

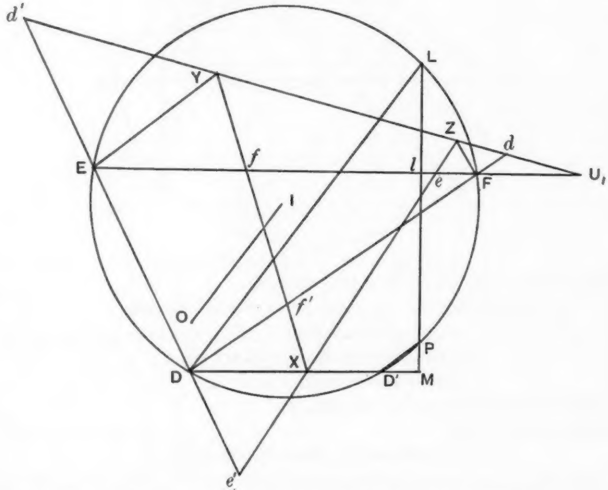
so that $Ff = Ff' = \tfrac{1}{2}c$.

Let L, L_2, L_3 denote the Simson line of P with regard to

(1) EDF in the nine-point circle;

(2) Eee' in the circle $EYeP$;

(3) XYZ in the in-circle.



Since DEF, Eee' have two common sides, L_1 coincides with L_2.

And since Eee' and XYZ have a common side XZ,

$\therefore$ L_2 and L_3 have a common point on XZ, and similarly they have a common point on YZ and on XY.

$\therefore$ L_1, L_2, L_3 coincide.

Two sides of DEF with one side of XYZ determine nine triangles.

Two sides of XYZ with one side of DEF determine nine other triangles.

These eighteen triangles, with DEF and XYZ, give a system of twenty circum-circles passing through P.

The twenty orthocentres, together with I, lie on OI, which is therefore a Twenty-one-Point line.

The parabola described with focus P and directrix OI will touch the sides of DEF and XYZ: and a ray of light passing along OI and incident on any one of the six tangents will be reflected to P.

6. If OI meets the circum-circle of ABC in Q and R.

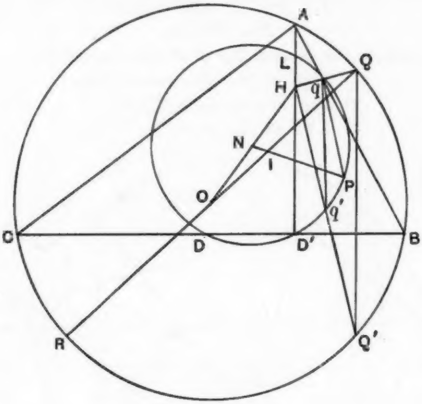
To prove that the Simson lines of Q and R with regard to ABC pass through P.

Draw QQ' perpendicular to BC, then AQ' is parallel to the Simson line of Q.

From the similar figures $HAQQ'$, $HLqq'$

Lq' is parallel to AQ', *i.e.* to the Simson line;

and qq' is parallel to AD', so that arc Lq = arc $q'D'$.



Now $\quad PDD' = AOQ$ (par. 4)

$\quad = LNq$, from similar figures,

$\quad = 2 \,.\, LD'q$;

$\therefore$ arc $PD' = 2 \,.$ arc $LQ = 2 \,.$ arc $D'q'$;

$\therefore$ arc Pq' = arc Lq;

$\therefore$ Pq is parallel to Lq' or to AQ'.

It also bisects HQ,

$\therefore$ Pq is the Simson line of Q.

7. The nine-point circle and ex-circles.

To find P_3 the point of contact of the nine-point and C ex-circle; produce DE to e_1, making $Ee_1 = Ee = \frac{1}{2}b$.

e_1e will cut BC in X_3 and AB in Z_3, points of contact of the ex-circle.

Let e_1e cut DF in V.

Bisect the arc $DD'F$ in β', so that β' is diametrically opposite to β.

Draw $\beta' V$ cutting the nine-point circle in P_3, and AC in Y_3.

Then it may be shown, as before, that Y_3 is a point of contact of the ex-circle; the tangents at Y_3 and β' being parallel, and P_3 a point of similitude for the two circles, which therefore touch at P_3.

The angles DP_3V, FP_3V on equal arcs are equal.

$$\therefore \; DP_3 : FP_3 = DV : VF = DX_3 : Fe \text{ or } FZ_3$$
$$= b + c : a - b.$$

Also $\quad DP_3 \,.\, \tfrac{1}{2}a = EP_3 \,.\, \tfrac{1}{2}b + FP_3 \,.\, \tfrac{1}{2}c,$

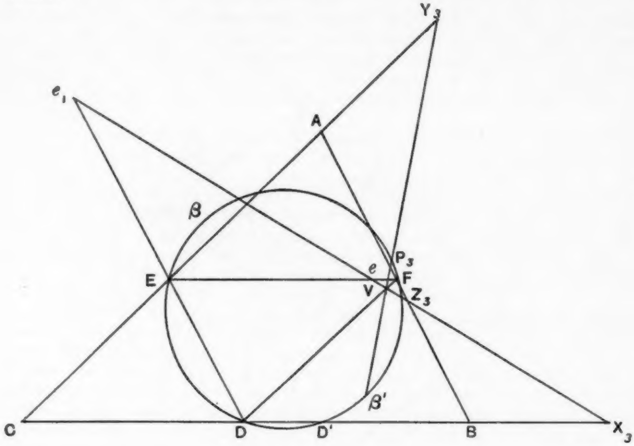
and $\quad (b+c)a = (c+a)b + (a-b)c;$

$$\therefore \; DP_3 : EP_3 : FP_3 = b + c : c + a : a - b;$$

$$\therefore \; FP_3 = EP_3 - DP_3.$$

If FE is produced to e_2, so that $Ee_2 = \frac{1}{2}b$, the circum-circles of Ee_1e_2, $e'Ee_2$ will cut the N.P. circle in P_2 and P_1 respectively.

It is noteworthy that $PP_1P_2P_3$ can thus be found independently of ABC or XYZ.



8. As YP passes through U_2, bisecting DPF,
so XP „ „ U_1, „ EPF (externally),
ZP „ „ U_3, „ DPE (externally).

It is easily seen that for U_1 the trilinear coordinates are given by

$$a\alpha : b\beta : c\gamma = -(b-c) : c-a : a-b,$$

for U_2 the ratios are $b-c : -(c-a) : a-b$,
„ U_3 „ „ $b-c : c-a : -(a-b)$.

Whence it follows that

$$AU_2U_3,\ BU_3U_1,\ CU_1U_2 \text{ are straight lines,}$$

and that AU_1, BU_2, CU_3 are parallel.

The isogonals to these parallels meet at the point S on the circum-circle, the coordinates of S being given by $\alpha : \beta : \gamma = \dfrac{a}{b-c} : \dfrac{b}{c-a} : \dfrac{c}{a-b}$.

S is the focus of a parabola touching the sides of ABC, and having its axis parallel to AU_1, BU_2, CU_3.

If AU_2U_3 cut BC in T_1, and CU_2U_1 cut AB in T_3, then

$$BT_1 : CT_1 = a-b : a-c,$$

and

$$AT_3 : BT_3 = a-c : b-c.$$

Hence the equation to T_1T_3 is

$$\frac{a\alpha}{b-c} + \frac{b\beta}{c-a} + \frac{c\gamma}{a-b} = 0,$$

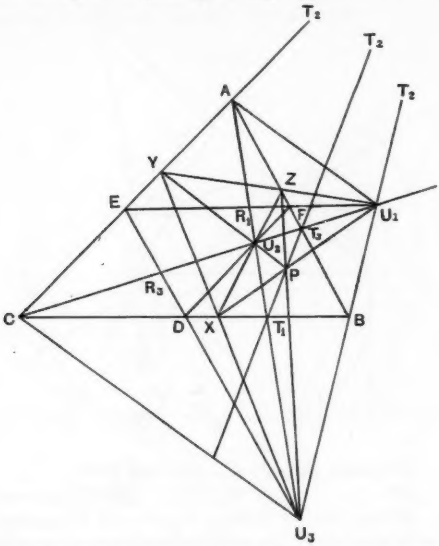
so that T_1T_3 is the tangent at P.

Thus the common tangent $T_1T_2T_3$ can be drawn and its point of contact P determined independently of the nine-point and in-circle.

This tangent divides each of the parallels AU_1, BU_2, CU_3 in the ratio 2 : 1.

9. $XYZP$, CAT_3T_1 being associated in- and circum-quadrilaterals for the in-circle, the pairs of diagonals YP, ZX and AT_1, CT_3 have the common

point of intersection U_2; and the mid points $R_1R_2R_3$ of the diagonals of CAT_3T_1 lie on a straight line passing through I.



The equation of $R_1R_2R_3$ is found to be $a(b-c)\alpha+b(c-a)\beta+c(a-b)\gamma=0$, so that it passes also through G.

The diagram shows that $U_1U_2U_3$ is a self-conjugate triangle for the in-circle, and that therefore its orthocentre is I.

Since PU_1 is the external bisector of EPF,

$$\therefore\ EU_1 : FU_1 = EP : FP = a-c : a-b = CT_1 : T_1B = ER_1 : R_1F\,;$$

$$\therefore\ PR_1 \text{ is the internal bisector,}$$

and U_1PR_1 is a right angle. W. Gallatly.

[In view of the coincidences which occur so frequently in mathematical research, it may be well to state that this paper is entirely original. W. G.]

244. [J. 1. a. γ.] *The number of Homogeneous Products.*

Any one of the homogeneous products of degree 7 of the four letters a, b, c, d, may be expressed in a special notation which will be found convenient.

$a^3\ b\ \ d^2$ is denoted by ...|.||..

$a\ \ b^2\ c^4$,, ,, .|..|....|

The strokes divide the dots into groups; the dots in the first group stand for a's and so on. The number of products is the number of ways of arranging the $7+\overline{4-1}$ symbols of which 7 are dots and $\overline{4-1}$ are strokes.

$$\therefore\ {}_4H_7=\frac{\lfloor\underline{7+4-1}}{\lfloor\underline{7}\ \lfloor\underline{4-1}}$$

In the same way it can be shewn that

$${}_nH_r=\frac{\lfloor\underline{n+r-1}}{\lfloor\underline{n-1}\ \lfloor\underline{r}}$$

F. J. W. Whipple.

245. [R. 4. a. α.] *A proof of the Theorem that the vector-sum of the 'axes' of two couples is the 'axis' of their resultant.*

The 'axis' of a couple is a vector drawn from any point in a direction normal to the plane of the couple, towards that side of the plane viewed from which the couple appears positive, and representing by its length the magnitude of the couple's moment. We assume as already proved the equivalence of all couples whose axes are equal in magnitude and the same in direction.

It is to be noted that if $ABCD$ be a rectangle of which the sides AB, CD represent two unit forces, then these forces form a couple whose axis is got by rotating BC about AB in the negative direction as seen from A. Let now AB be part of the line of intersection of the planes of two couples whose moments are G_1 and G_2, and let it represent a unit force. Let the rectangles $ABCD$, $ABEF$ be completed in the planes of G_1 and G_2 respectively, in such a way that the forces AB and CD give the couple G_1 and the forces BA and FE give the couple G_2. This implies, since these are unit forces, that $BC=G_1$ and $EB=G_2$.

Since the forces AB and BA balance one another, it is clear that the two couples G_1 and G_2 are equivalent to a single couple whose forces are CD and FE, and whose moment is therefore measured by EC.

If now the triangle EBC be rotated about AB as axis through $-90°$ as seen from A, it is clear that if $E'BC'$ be its new position, then BC', $E'B$ and $E'C'$ will be the 'axes' of the couples G_1 and G_2, and the resultant couple respectively. But $E'C'$ is the vector-sum of $E'B$ and BC'. Thus the proposition is proved. R. F. MUIRHEAD.

246. [K. 9. b.] *Note on a Chinese Theorem.*

In the *Gazette*, Vol. III., p. 269, the Editor called attention to a geometrical theorem due to a Chinese mathematician. "If $A_1 \dots A_n$ is a polygon inscribed in a circle and S_r is the sum of the in-radii of the triangles which have A_r for a common vertex and the sides of the polygon (except those on which A_r lies) as sides opposite A_r, then $S_1=S_2=\dots=S_n$.

A simple expression for this sum may be obtained by means of the following lemma, the proof of which will be left to the reader.

Lemma. If p_1, p_2, p_3 are the perpendiculars from the centre of the circumscribing circle on to the sides of a $\triangle$ and R, r are the radii of the circumscribed and inscribed circles, then

$$p_1+p_2+p_3=R+r.$$

Applying this lemma to the different triangles having A_1 as vertex, noticing that the perpendicular on A_1A_s must be regarded as positive for one triangle having A_1A_s as side and negative for the other, we obtain

$$(n-2)R+\Sigma r_k=\Sigma p_k$$

=sum of the perpendiculars from the centre of the circumscribing circle on to the sides of the polygon. H. BATEMAN.

247. [C. 2. a.] *Integration of* $\dfrac{1}{\sqrt{x^2-a^2}}$ *and* $\dfrac{1}{\sqrt{x^2+a^2}}$.

In $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}}$ put $\sqrt{x^2-a^2}=y$, then $x^2-a^2=y^2$, thus $xdx=ydy$ and

$$\frac{dx}{\sqrt{x^2-a^2}}=\frac{dx}{y}=\frac{dy}{x}=\frac{dx+dy}{x+y}$$ (by adding numerators and denominators);

therefore $$\int\frac{dx}{\sqrt{x^2-a^2}}=\log(x+y)=\log(x+\sqrt{x^2-a^2}),$$

and since $$x=\sqrt{y^2+a^2};$$

$\therefore$ also $$\int\frac{dy}{\sqrt{y^2+a^2}}=\int\frac{dy}{x}=\log(x+y)=\log(\sqrt{y^2+a^2}+y).$$ G. H. BRYAN.

248. [C. 2. J.] *The Solution of the "Christmas Cake" Problem.*

The problem of dividing a circular cake in the manner suggested in *Nature* for December 20, so that the areas of the portions eaten on three consecutive days shall be equal, is easily solved correct to any required number of decimal places by the following or some such method :

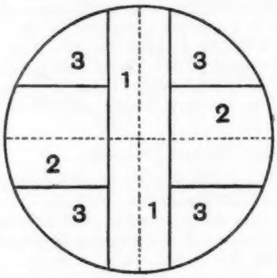
Let x be the ratio of the thickness of the first slice to the diameter of the cake, then since the area of the first slice is one third of that of the cake, we easily obtain

$$\frac{\pi}{12}=\frac{x\sqrt{(1-x^2)}}{2}+\frac{1}{2}\sin^{-1}x=\int_0^x\sqrt{1-x^2}\,dx.$$

Since x is a proper fraction, the easiest plan is to use the second form and expand the quantity under the integral in a convergent series, giving, for x, the equation,

$$\frac{\pi}{12}=x-\frac{1}{2}\frac{x^3}{3}-\frac{1.1}{2.4}\frac{x^5}{5}-\frac{1.1.3}{2.4.6}\frac{x^7}{7}\dots$$

or $$3{\cdot}14159\dots-12x+2x^3+0{\cdot}3x^5+0{\cdot}103\dots x^7+\dots=0.$$



This is easily solved by Horner's method. The trial divisor gives immediately the first digit 2.

To find the ratio of the breadth of the second pair of slices to the diameter of the cake, we notice that the height of the slices has to be diminished by the portion removed in the first operation. If x_1 is the root of the previous equation, we must subtract x_1x from the right-hand side of the first equation, and we get as our new equation for x,

$$3{\cdot}14159\dots-12(1-x_1)x_2+2x_2^3+0{\cdot}3x_2^5+\dots=0.$$

The results I obtain are $x_1={\cdot}26407420\dots$ and $x_2={\cdot}367180\dots$, which might easily be checked by recalculation, as the working is simple.

G. H. Bryan.

QUERIES.

(15) Where shall I find the best treatment of "Linkages" suitable for boys of 15-16? "Linkages" is set as part of the London University Syllabus for Applied Mathematics, Inter-Sci. A.

(16) Given a curve in which each abscissa represents the space described from a given time by a moving particle under the action of forces represented by the corresponding ordinates; how may the change in kinetic energy of the particle be expressed geometrically? δK.E.

(17) Under what conditions may the locus of the corners of a quadrilateral, circumscribed to an ellipse E, consist of two ellipses confocal with E?

Student.

(18) From a fixed point T, tangents TA, TB are drawn to a conic whose centre O is fixed. As the conic rotates about O, to find the loci of the remarkable points of the triangle TAB. Can the degree of any of the loci be determined by *à priori* consideration? INQUIRER.

(19) What is the largest prime number known? Where can I get information as to the frequency of primes? a.

(20) Have e^e and π^π been developed in series? EPPE.

(21) The works on analytical conics which I have at my disposal seem to deal with polar coordinates in a very perfunctory way. Does any work treat of conics in polar coordinates, as we find books on trilinears, and, if not, is it because of any special algebraical difficulties that arise in "polar" treatment? P.C.

(22) Can "graphs" be utilised in the solution of problems of probability? ?

(23) Wanted a formula of approximation for $n!$ when n is very large. !

ANSWERS TO QUERIES.

[6, p. 95.] The statement is in *Messenger*, Vol. XVI., p. 21; see also Vol. XXIX., p. 22.

We can find x, y, z, ω, so that

$$A\alpha = x + y\alpha + z\alpha^2 + \omega\alpha^3, \text{ etc.},$$

where

$$A = (a-b)(a-c)(a-d), \text{ etc.}$$

Now

$$\Sigma a^r/A = 0 \text{ if } r = 0, 1, 2. \qquad (1)$$

Hence

$$\omega = 0,\ z = 0,\ A\alpha = x + y a, \text{ etc};$$

$$\therefore\ \Sigma A^p \alpha^{p+1} a^q = \Sigma A^{-1} a^q (x + ya)^{p+1}.$$

Making in succession $p=0$, $q=2$; $p=1$, $q=0$; $p=1$, $q=1$; $p=2$, $q=0$, we have by (1)

$$\Sigma a^2\alpha = y,\ \Sigma A\alpha^2 = 0,\ \Sigma A a\alpha^2 = y^2,\ \Sigma A^2\alpha^3 = y^3,$$

whence the required results follow. E. J. NANSON.

[8, p. 95.] ἐπί asks for mnemonics for the first few decimal places of π; he does not define "few," but if he can remember 3·14159, the next 5 places may be recovered by adding to the decimal part the terms of the series 1, 2, 4, 8, 16 neglecting all carrying, giving for the next five figures:

$$1+1=2;\ 4+2=6;\ 1+4=5;\ 5+8=\ 3;\ 9+16=\ 5.$$

Wellington College, Berks. S. A. SAUNDER.

[9, p. 95.] It is to be noticed that if a, b, c, d be the sides of any convex quadrilateral, e the diagonal (ad, bc), and f the diagonal (ab, cd), then

$$2(ab+cd)(ac+bd)(ad+bc) = e^2(ab+cd)^2 + f^2(bc+ad)^2 + \{(ac+bd)^2 - e^2f^2\}(a^2+b^2+c^2+d^2-e^2-f^2),$$

the last term on the right vanishing when the quadrilateral is cyclic or parallelogrammic; and it would be interesting to know whether this relation, with which "Gemmi" is evidently acquainted, is to be found in any text book.

Now $e^2(ab+cd)^2 + f^2(bc+ad)^2 > 2ef(ab+cd)(bc+ad)$.

Let this be substituted, and after division by $ac+bd-ef$ the inequality stated easily follows.

The fact that a, b, c, d, e, f are not all independent but bound together by a fixed relation probably explains the paradox:—

The general quadrilateral is cyclic if $(a^2+f^2-d^2)/2af = (c^2+e^2-d^2)/2ce$; but this does *not* involve the factor $ac+bd-ef$ as might *a priori* be expected.

Q.

[10, p. 95.] The example is due to Genese and the explanation to Cayley, see *Messenger*, Vol. VII., pp. 61-63. The rule for finding an envelope does not lead to an envelope unless the two-fold value of the parameter is variable, not constant. Here the values of λ corresponding to the factors x, y, $x/a+y/b-1$ are 0, ∞, a/b and are constant, not variable. E. J. NANSON.

[10, p. 95.] *Q.*'s mistake lies in his misinterpretation of the equation $xy\left(\frac{x}{a}+\frac{y}{b}-1\right)=0$. It is true that this may mean the three lines $x=0$, $y=0$, $\frac{x}{a}+\frac{y}{b}-1=0$. But it may also mean, and in this case it certainly does mean, the three points $y=0$, $\frac{x}{a}+\frac{y}{b}-1=0$; $\frac{x}{a}+\frac{y}{b}-1=0$, $x=0$; $x=0$, $y=0$.

As all the parabolas pass through these 3 points, we are not surprised at finding them occur in the envelope; the other factor is $1=0$, the line-infinity, which all parabolas touch. W. D. EVANS.

King's College, Cambridge.

[14, p. 95.] Let the graph of $y=a^x$ be drawn and all trace of the y axis, a, and the unit used be removed. Taking an arbitrary position for new y axis, it is readily seen that the graph has for equation $y=b^x$ where b is the value of y when $x=1$, the new unit being the value of y when $x=0$.

The graph manifestly has unit slope at some point. Take the origin below this point and denote the corresponding value of b by e. Measurement of the ordinate where $x=1$ gives $e=2{\cdot}7$, and from the definition of a tangent it follows that

$$\underset{x=0}{Lt}(e^x-1)/x=1.$$

Hence the derivatives of e^x, $\log x$ are found, and by the method given, Vol. III., p. 238, we can readily approximate to e as closely as we please.

E. J. NANSON.

REVIEWS.

Elements of Descriptive Geometry. By O. E. RANDALL, Ph.D. Ginn & Co.

This is a scholarly treatise and on a subject much neglected by English mathematical teachers. The neglect is no doubt mainly due to the examination system, which has not even required the principles of the art to form part of the school course. The compilation of text-books has therefore fallen mainly into the hands of writers who have treated the subject only from the point of the Technical Class or the South Kensington Examinee, and have produced medleys of such snippets of Practical Geometry and other subjects as may be necessary to obtain a pass in the First or Second Stage. It might easily happen that a man "with a good mathematical degree" would be puzzled if called upon to give an account of the methods of Descriptive Geometry. Professor Henrici, in his address as President of Section A (Southport, 1883), attributed this and the neglect of other departments of Pure Geometry to the "grasp of the dead hand" of Euclid: "Most of all, perhaps, solid geometry has suffered, because Euclid's treatment of it is scanty, and it seems almost incredible that a great part of it—the mensuration of areas of simple curved surfaces and of volumes of simple solids—is not included in ordinary school teaching, . . . *and what is almost worse is that the general relation of points, lines, and planes in space is scarcely touched upon, instead of being fully impressed on the student's mind. The methods for doing this have long been developed in the new geometry which originated in France with Monge. But these have never been thoroughly introduced.*" Twenty years later the hold of Euclid was relaxed, and the complaint of Professor Henrici about the non-inclusion of the mensuration of the simpler solids can no longer be made, but Monge's method of investigating relations of points, lines, and planes in spaces

has not yet found its way into the ordinary text-book. Without wishing to see Descriptive Geometry introduced formally into the examination syllabus, we would like to see the simple theorems on which his practical constructions depend appear in connection with the fundamental ones of a school course, to which they would form easy exercises. The rotation of one plane into coincidence with another might easily be introduced as a common device for the practical solution *in plano* of simple trigonometrical problems on heights and distances in three dimensions. But while it is not at all desirable that Descriptive Geometry should be formally introduced into the school course, it is highly desirable that teachers of Geometry should familiarise themselves with Monge's methods, and that, if they cannot obtain a copy of his *Géométrie Descriptive*, they should have recourse to some other continuous treatise, such as the one before us, on "that great alphabet of the application of geometry to the arts." If *seri studiorum* find the unwonted diagrams at first rather troublesome to see through, we can strongly recommend T. Jones' set of models (Heywood, 2s.) or J. Schotke's set of 30 stereoscopic slides (L. Friederichsen & Co., Neuerwall 61, Hamburg), if the latter are still to be obtained.

About 70 pages are devoted to the Point, Line, and Plane, 120 to curves and curved surfaces, and the last 19 pages to one-plane methods of projection, of which the isometric is treated most fully. Principles are clearly laid down, and constructions are preceded by the analysis which led to them. Chapter VIII., on the Generation and Classification of Surfaces, does not seem quite satisfactory. It is not pointed out that the same surfaces may be generated in more than one way, and words are used on p. 88 which seem to imply that all surfaces are developable.

Mathematical Drawing. By G. M. Minchin, M.A., F.R.S., and J. B. Dale, M.A. (Arnold.)

Graphs. By C. H. French, M.A., and G. Osborn, M.A. (W. B. Clive.)

Messrs. Minchin and Dale have supplied a distinct and widely felt want. The contents of their book may be regarded as the higher development of the schemes of practical work now included in school courses of geometry. Its four parts are devoted respectively to: I. Graphic multiplication, Approximate rectification of circular arcs, Amsler's planimeter; II. The constructions for various problems on conic sections by metric methods; III. Various curves other than conics and their connection with the solution of equations; IV. Conical projection. The third and fourth sections are the most interesting, dealing as they do with matters rather out of or beyond the ordinary school text-book, but the other parts contain much that is interesting, especially the clear treatment of the theory of Amsler's planimeter and its applications to finding centres of gravity and radii of gyration of a closed contour. In section III. the curves discussed are mostly those which arise from the graphic representation of the results of physical investigations. The treatment is thorough, and sufficient examples are solved in detail to enable the student to apply the same methods to analogous cases or to develop them further. There is a sub-section on "Dipolar Co-ordinates" which might profitably be read in connection with Professor Genese's "Biangular Co-ordinates" in Milne's *Companion.* Among the curves selected for special treatment are the Catenary (which has about 10 pages to itself), the Magnetic Curve, Bernouilli's Lemniscate, and the Ovals of Descartes and Cassini. Useful as the previous three sections are, we believe the fourth—on the application to practical constructions of the methods of Projective Geometry—to be the most valuable contribution to the improvement of the higher work of schools and to the progress of the right sort of technical mathematics that has been recently made. For the higher parts of the theory the authors naturally refer their readers to the standard treatises on the subject, but we believe that the student could scarcely have a better introduction to the treatises of Russell or Cremona. Few subjects lend themselves more easily to the interweaving of theory and practice than this, and the authors have wisely given a connected view of the theory as far as it applies to the constructions used. The work will be widely welcomed by one important class of students, viz., those who take Mathematics as one of their subjects for the B.A. examination of the University of London, and especially those external students whose isolation makes it difficult or impossible to

attend a good course of lectures. It is just what they want. But we recommend it to all teachers who have to plan courses of practical work for the schools whose mathematics are under their charge. It forms an excellent finish for the school course to lead up to.

Believing that a second edition is likely to be wanted before long, we venture to call attention to the following points in which we think there is room for improvement:

(i) A few extra constructions might be given in Part II., especially some well-known ones for the parabola and hyperbola.

(ii) A few notes might be added to Part IV. pointing out how some of the metric constructions, although readily proved by Euclidean methods, follow still more readily from the Projective Properties given. The Boscovich construction, for instance, might be shown to depend on the transformation of a conic into a circle, and its connection with Focal Projection pointed out.

(iii) A few analytical notes would be welcome. It might be pointed out how, by simple manipulation of an algebraical equation, a variety of geometrical constructions are suggested. *E.g.* the method suggested by the equation

$$(x-y)^2=8(x-2)$$

is a modification of the metric method on p. 17 by means of the lines $x=y$, $x=2$. But by writing the equation

$$y(y-2x)=(x-4)^2,$$

we see that we have a case of example 1 on p. 123.

No rearrangement or dislocation would be necessary. Most of what we think advisable might come in at the end with additional exercises.

Messrs. French and Osborn describe the subject of their little book in the sub-title as *The Graphical Representation of Algebraic Functions.* It also may be regarded as a further development of school graph work. Introducing the subject by means of two chapters on plotting statistics and arithmetical problems, they lose little time in getting to its more important bearings on algebraical theory. It has been developed out of the smaller work previously written under the same title by the addition of matter required by more advanced students. The tracing of curves and its application to approximate solution of equations is extended not only to various forms of the equation of the second degree, but to some of the third and fourth. The explanations are very thoroughly done, and there is no stint of useful diagrams. In a chapter on "Slope" the process of differentiation is well illustrated and explained. Like the work on *Mathematical Drawing* noticed above, it promises to be very useful to students presenting themselves at the Final Arts Examination of the London University, and is in some ways complementary to it. They may advantageously use it as a companion volume to their Smith's *Conics* or Loney's *Coordinate Geometry.* We think that the algebraical and geometrical work might with advantage be still more closely interwoven, and that more attention should be given to the interpretation, though either of results obtained through the other. In finding points on a conic whose equation is given, there are many cases in which a few obvious points having been found arithmetically others might be obtained by ruler constructions much more readily than by arithmetical approximation and subsequent plotting. We looked through the chapter on Maxima and Minima without finding a word on the connection between its problems and those of "tangency" and "equal roots." It would help the student to point out that when he writes $y=(x-3)^2+2$ for $y=x^2-9x+11$, that since $y=2$ gives two equal values 3, 3 for x, it is the tangent at (3, 2), especially as the method is so easily extended. *E.g.* writing $y=x^2$ in the form $y=(x-p)^2+2px-p^2$ we see that $y=2px-p^2$ touches at (p, p^2), and the figure on p. 97 might have been justified by writing $y=(x+1)^2(x-2)+3x+2$ for $y=x^3$.

Projective Geometrie in synthetischer Behandlung. Von Prof. Dr. Karl Doehlmann. G. J. Göschen, Leipzig.

On a previous occasion we have had the pleasure of reviewing a book of the "Sammlung Goschen" Series, and we use the word pleasure in a real and not a conventional sense. Here we have at the price of about tenpence a compact treatise

on Projective Geometry of about twice the length of Prof. Henrici's article on that subject in the *Encyclopædia Britannica*. We could wish that some similar work existed in English. We have, of course, longer and deeper treatises, and we have slight sketches useful in their way in books on Analytical Geometry, but we know of nothing except Prof. Henrici's article, which cannot of course be obtained separately, which has the same aim as the little work before us. The author expounds successively the Principle of Duality, the Point, Line and Plane at Infinity, Anharmonic Ratio, and Involution. He then treats the Conic systematically as the locus or envelope of the "element" arising from the combination of two "corresponding elements" of a one-to-one correspondence. Having next treated of Pole and Polar, he concludes with a section on the Ruled Surface as generated in a manner analogous to the conic from two projective "fundamental forms." We have found the treatment very clear and stimulative. In spite of the small space at his disposal, the author finds room for 91 excellent figures, special cases of Pascal and Brianchon each having four allotted to them. Excellent practical constructions are given and shown to follow readily from the general theorems. Going over the same ground as the section on Projection in the work on Mathematical Drawing noticed above, it might profitably be read concurrently with it. If the other 24 volumes of the "Kleine mathematische Bibliothek" are of the same high merit as the *Projective Geometrie*, they form a valuable collection.

Auslese aus meiner Unterrichts- und Vorlesungspraxis. Von Prof. Dr. Hermann Schubert. II. and III. G. J. Göschen, Leipzig.

The interesting contents of Vol. I. have already been noticed. The three sections of Vol. II. deal respectively with (1) Areas of Triangles and Inscribed Polygons, (2) Continued Fractions and the Theory of Numbers, (3) The Computation of Logarithms without the use of Logarithmic Series. In (1) are treated the cases in which Hero's formula for the area of a triangle gives an integral result, and a table of such Hero's triangles is given. The author then extends his investigations to cases in which a median or the bisector of an angle is rational. The reader may perhaps like to set his pupils to verify the author's statement that if the sides of a triangle are 127, 131, 158 the medians are halves respectively of the numbers 261, 255, 204, and to use the numbers here given to illustrate the known theorem that if the sides of one triangle are proportional to the medians of another the relation is a reciprocal one. Similar results for cyclic polygons and certain pyramids are discussed and tables given of Hero's Cyclic Quadrilaterals. In (2) the properties of numbers are investigated by means of the convergents of continued fractions. Much attention is given to the representation of various numbers as the sum of two square numbers. As a specimen of the numerical results obtained, we select the following:

$$146375125 = 11606^2 + 3417^2 = 11577^2 + 3514^2 = 12066^2 + 887^2 = 12073^2 + 786^2.$$

In (3), by methods analogous to a similar section in Vol. I., we are shown how to obtain 4 place logarithms. *E.g.* by means of 4 equations like

$$2 \log 61 + \log 31 - \log 11 - 20 \log 2 + 2 = \cdot 000031\ldots,$$

the logarithms of 61, 31, 11, and 2 may be obtained by elimination.

The five sections of Vol. III. deal respectively with (1) the position of the C.G. of various plane and solid figures, (2) the parabola in elementary analytical geometry, (3) Snell's law, (4) the volume of a prismatic frustum, (5) the extension of Simpson's formula, (6) the formulae of spherical trigonometry, (7) "Heronic" spherical triangles. All these matters are treated in an interesting manner, and the sections are just such as might have been contributed to the *Gazette* in the spirit of the announcement in its original prospectus, "It cannot be doubted that many teachers are in possession of methods of their own which experience has shown to be better than those most in vogue. They are asked to let others have the advantage of knowing these special methods." Dr. Schubert is to be congratulated on having made such an interesting collection. Perhaps the reason that we now get fewer of such contributions to the *Gazette* is that they are kept in desks and pigeon-holes in the hope of forming similar volumes to his.

Guido Hauck. By E. Lampe and A. Parisius. Teubner, Leipzig.

These are memorial addresses dealing with the life work and character of a famous teacher of technical mathematics in their highest development. He held high office in Tübingen, Berlin, and Stuttgart, and seems throughout his career to have gone deeply into the principles underlying technical work, notably in his treatment of Descriptive Geometry and of Perspective. In the latter he dealt not only with the ordinary "Glastafele" theory, but with its limitations in artistic work, and with photogrammetry.

Text-Book on the Strength of Materials. By S. E. Slocum and E. L. Hancock. Pp. xii, 314. Ginn & Co. 12s. 6d.

The authors of this valuable addition to the library of the engineering student have borne in mind the necessity of providing for the concurrent study of the theoretical and practical aspects of the strength of materials. Here will be found certain matter to be discovered in no text-book with which we are at present acquainted—we may mention the chapter on reinforced concrete as an instance in point. The theoretical side is treated in Part I., and the experimental in Part II. Chapters on the elastic properties of materials and the fundamental relations between stresses and deformations are followed by a discussion of the analysis of stresses in beams. Here among the novelties in the book we may refer to a graphical method of finding the moment of inertia and centre of gravity of a rail-shape, due to Nehrs, and of a non-homogeneous section such as that of a reinforced concrete beam. In the next chapter we find a treatment of the flexure of built-in beams in reinforced concrete construction. After a section on Clapeyron's theorem of three moments we are given Föppl's proof of Maxwell's theorem as applied to the theory of continuous beams, with a statement of Castigliano's theorem and its application to the same problem. The chapters on columns and struts are of the ordinary content, but that on spheres and cylinders under uniform pressure presents a novelty in the formula, giving the critical external pressure just preceding collapse of a hollow circular cylinder. Lamé's formulas for thick cylinders readily follow. Chapters VIII. and IX., dealing with the analysis of stress in flat plates and hooks, links, and springs, are based on Bach and Résal, and are as up-to-date as is possible in the present unsatisfactory state of this branch of the subject. The theoretical portion concludes with chapters on arches, arched ribs, foundations and retaining walls. The practical portion of the volume deals with the physical properties of materials, the typical testing machines, and the usual apparatus for measuring stresses. Here the tables given will be found of use to the student. The selection of problems throughout seems to us to be unusually happy, and great credit is due to the authors for the admirably simple exposition they have given of a subject in which it is not always easy to be both clear and concise.

ERRATA.

P. 103, l. 23, for 'viel' read 'viele.'

P. 104, l. 2, omit second 'we.'

l. 5, for 'on' read 'ont.'

FOR SALE.

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